THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

29 AUG 1924

EDITED BY

M. T. NARANIENGAR, M.A.

WITH THE CO-OPERATION OF

Dr. R. P. PARANJPYE, M.A., D.Sc.,

Prof. A. C. L. WILKINSON, M.A., F.R.A.S.,

and others.

Volume XV, No. 9.

June 1924.

1593

MADRAS:

PRINTED BY SRINIVASA VARADACHARI & CO.

Annual Subscription, Rs. 6.] **[Single Copy, One Rupee.**

CONTENTS.

	Page
Invariant-Factors and Integer-Sequences: R. Vaidyanathaswamy, M.A., F.R.S.E.	189
Twin Points: M. Bhimasena Rao	199
Definition of Straight Line: S. V. Ramamurthy, M.A., I.C.S. ...	211
NOTES AND QUESTIONS:	
Note on the representation of a Number as the sum of two others satisfying given conditions: A. A. Krishnaswami Ayyangar	*137*
A Note: Hans R. Gupta	*140*
Solutions	*141*
Questions for Solution	*151*

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

The Journal is open to contributions from members as well as subscribers. The Editors may also accept contributions from others.

Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper and all diagrams should be neatly and accurately drawn on separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary and Editor M. T. NARANIENGAR, M.A., 24, Mallesvaram, Bangalore, or Joint Editor K. ANANDA RAO, M.A., Professor, Presidency College, Madras.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secy. G. A. SRINIVASAN, M.A., Presidency College, Triplicane, Madras.

Enquiries by intending members of the Society and all other communications may be addressed to the Hony. Joint Secretary P. V. SESHU AIYAR, B.A., L.T., Principal, Government College, Kumbakonam.

Errata Slip.

In Vol. XV, No 8, Part I.

	For	*Substitute*
p. 183 line 4	G. T. Taylor	G. I. Taylor
p. 183 ,, 25	,,	,,
p. 184 ,, 2	showed	shows
p. 184 ,, 22	axis	axis of x
p. 184 ,, 25	A_2	$-A_2$
p. 184 ,, 31	Jeffrey	Jeffreys
p. 185 ,, 21	Jeffrey's	Jeffreys'
p. 185 ,, 23	which	the solution of which
p. 186 ,, 20	speed	shear
p. 186 ,, 22	ρ	$\rho\lambda$

In lines 4 and 5 of *Question 1327* (p. 135 of Vol. XV, No. 8, Part II: *Notes and Questions*).

read $\tan^{-1}(\mathrm{sc}\, u . \mathrm{dn}\, v)$ *for* $\tan^{-1} \mathrm{sc}\, u . \mathrm{dn}\, v$.

,, $\tan^{-1}(\mathrm{sc}\, v . \mathrm{dn}\, u)$,, $\tan^{-1} \mathrm{sc}\, v . \mathrm{dn}\, u$.

,, $\tan^{-1}(k\, \mathrm{sd}\, u . \mathrm{cn}\, v)$,, $\tan^{-1} k\, \mathrm{sd}\, u . \mathrm{cn}\, v$.

,, $\tan^{-1}(k\, \mathrm{sd}\, v . \mathrm{cn}\, u)$,, $\tan^{-1} k\, \mathrm{sd}\, v . \mathrm{cn}\, u$.

Invariant-Factors and Integer-Sequences.

By R. Vaidyanathaswamy, M.A., F.R.S.E.

The theory of characteristic (or 'latent') roots of a matrix is perfect smooth-sailing when they are all distinct. But when a characteristic root α is repeated a certain number of times—say r times, then a little reflection will suffice to show that the properties of the matrix in relation to the characteristic root α are *not* completely determined by a knowledge of the number r alone. That is to say, over and above the *fact* of coalescence of r characteristic roots, there is a *manner* of coalescence about which the number r by itself gives no information. To specify completely this 'manner' of coalescence, requires a sequence (generally a decreasing sequence) of integers.

There are two classical theories of the matrix, the *algebraic* theory of Weierstrass and the *geometric* theory of Grassman. In the former, the representative sequence of a multiple characteristic root α is the sequence of the indices of the *invariant-factors* corresponding to α: or briefly, the *invariant sequence* of α. In the latter, the matrix is regarded as the symbol of a collineation in hyperspace, and the representative sequence of α (which I propose to call the 'latency sequence') is derived from the orders of the latent and semi-latent regions of successive species for the 'latent' root α. The relation between the invariant sequence and the latency sequence is well-known. In this order of ideas, there arises naturally the problem of specifying the process by means of which one can derive from the representative sequences (that is to say, *either* the invariant *or* the latency sequences) of a common latent root α of two matrices Δ_1, Δ_2, the representative sequence of α with respect to the "direct product" of Δ_1 and Δ_2. It is clear that this process could be stated solely in terms of modes of combination of integer-sequences; that is to say, the matrix may be eliminated altogether, and the question may be treated as one on abstract decreasing sequences of positive integers. When this is done, the algebraic and the geometric theories will be put in very clear relation to one another.

We begin by treating of various relations and processes connected with *decreasing sequences of positive integers*. The results and ideas are then applied to the sequences which occur in the Weierstrassian and the Grassman theory.

§ 1. **Sequences.** By a 'sequence' we shall invariably understand (unless explicitly stated otherwise) a *decreasing sequence of positive integers*, that is, a sequence of the type

$$R = (r_0\ r_1\ \dots\ r_n)$$

where the r's are positive integers such that $r_k \geqslant r_{k+1}$. We shall also call the sum $r_0 + r_1 + \dots\dots + r_n$, the *weight* of the sequence R.

We shall find it useful to think that the sequence R consists, in addition to the r's of an infinite number of zero terms added at the end. From this point of view, every sequence R consists really of an infinite number of terms, of which however only a finite number at the beginning are greater than zero; and it is only for convenience that R is *denoted* by its non-vanishing part.

§ 2. **The Sum and the Compound.**

The *sum* of two sequences

$$R = (r_0\ r_1\ \dots\ r_n),$$
$$S = (s_0\ s_1\ \dots\ s_m),$$

is defined to be the (decreasing) sequence

$$R + S = (t_0\ t_1\ t_2\ \dots),$$

in which $t_k = r_k + s_k$ (r_k and s_k being considered to be zero, in accordance with our previous remark, when k is greater than n and m respectively).

The *compound* of the two sequences R and S is defined to be the sequence

$$R \oplus S = (t_0\ t_1\ \dots\ t_{m+n+1}),$$

where the t's consist of the r's and s's arranged in descending order of magnitude.

The sum and the compound of any number of sequences are defined in the same manner. From the definition it is clear that the weight of the sum or the compound of any number of sequences is the sum of their weights. It is also clear that the operation of taking the sum or the compound is commutative and associative.

§ 3. **The Conjugate of a Sequence.**

A sequence $R = (r_0 r_1 \dots r_n)$ of weight w equal to $(r_0 + r_1 + \dots + r_n)$ may be regarded as representing a special kind of partition of the number w, namely a partition into decreasing parts. Thus the sequence R may be represented graphically by $n + 1$ columns of dots, the kth column contain-

ing r_{k-1} dots and the top-dots of all the columns forming a horizontal row. If we turn this diagram through a right angle, we obtain the diagram of a new partition (called the *conjugate* of the original partition), also into decreasing parts of the same number w; and this new partition will correspond to a decreasing sequence S, which we define to be the *conjugate sequence* of R.

Two conjugate sequences R and S, where

$$R = (r_0\, r_1 \cdots r_n)$$
$$S = (s_0\, s_1 \cdots s_m),$$

are obviously related in this reciprocal manner, that $r_0 = m+1$, $s_0 = n+1$ and generally r_k = the number of s's which are greater than k, and s_k = the number of r's which are greater than k.

As an illustration, the sequence (5, 5, 4, 2, 1, 1) is represented by the graph

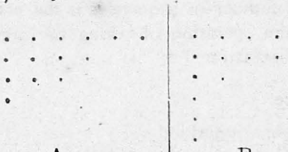
On turning the page through a right angle, the graph reads the partition corresponding to the *conjugate sequence* (6, 4, 3, 3, 2).

The three operations of taking the sum, the compound and the conjugate are connected by the following simple relation :—

THEOREM I. *The sum of the conjugates of any number of sequences is the conjugate of their compound*; or in symbols (denoting the conjugate of R by | R),

$$|\,R + |\,S + \ldots = |\,(R \oplus S \oplus \ldots).$$

The graphical proof of this theorem will become obvious by considering the case of two sequences A = (4, 3, 3, 2, 1, 1) and B = (6, 3, 2. 2).



A B

Fig. 1.

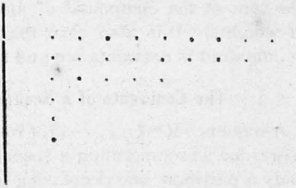


Fig. 2.

Place the graphs of A and B together as in Fig. 1. Then push every dot of B along the same horizontal line, to the left as far as it would go; we thus get Fig. 2.

It is obvious that Fig. 2 read by columns, gives the compound of the two sequences A and B. Now the sum of the conjugates of A and B is obtained by reading Fig. 1 by rows; but in obtaining Fig. 2 from Fig. 1, we have not altered the number of dots in any horizontal row. Hence the sum of the conjugates of A and B is also obtained by reading Fig. 2 by rows. Thus Fig. 2 read by columns gives the *compound*, and read by rows the *sum* of the conjugates of the two sequences, so that the conjugate of the compound is the sum of the conjugates.

§ 4. **The Derivate and the Integral.**

If $R = (r_0\ r_1\ \cdots\ r_n)$ be a decreasing squence, the sequence

$$\Delta R = R' = (r_0'\ r_1'\ \ldots\ r_n')$$

where $\quad r_0' = r_0 - r_1,\ r_1' = r_1 - r_2,\ \ldots\ r_n' = r_n)$

is a sequence of positive integers, but not necessarily a decreasing sequence If R' *is* a decreasing sequence, then R may be said to be *differentiable*, or to admit the *derivate* $\Delta R = R'$.

The *integral sequence* of R is defined to be the unique (differentiable) sequence the derivate of which is the sequence R. From the definition of the derivate, it follows that the integral of $R = (r_0\ r_1\ .\ .\ r_n)$ is given by

$$\Delta^{-1} R = (R_0\ R_1\ \cdots\ R_n)$$

where $\quad R_k = r_k + r_{k+1} + \cdots + r_n \quad (k = 0, 1 \cdots n).$

It may be remarked that a sequence $R = (r_0)$ containing only one term is its own integral and derivate.

§ 5. **The Product.**

Let there be two sequences R and S given by

$$R = (r_0\ r_1\ \cdots\ r_n)$$

$$S = (s_0\ s_1\ \cdots\ s_m).$$

Consider the sequence $T = (t_0\ t_1\ \cdots)$ defined by the relations:

$t_0 = r_0 + s_0$

$t_1 =$ the smaller of the numbers $r_0 + s_1,\ r_1 + s_0$;

$t_2 =$ the smallest of the numbers $r_0 + s_2, r_1 + s_1, r_2 + s_0$;

and generally $t_k =$ the smallest of the numbers $(r_\lambda + s_{k-\lambda})\ (\lambda = 0, 1 \ldots k)$.

The sequence T thus defined is necessarily a decreasing sequence. To prove this, assume that the least of the numbers $(r_\lambda + s_{k-\lambda})$ $(\lambda = 0$ $1, \dots k)$, is given by the value p_k of λ, p_k being an unknown function of k, Then we have :

$$\begin{aligned} t_{k+1} &= r_{p_{k+1}} + s_{k+1-p_{k+1}} \text{ (by definition of } t_{k+1}) \\ &\leqslant r_{p_k} + s_{k+1-p_k} \text{ (by definition of } p_{k+1}) \\ &\leqslant r_{p_k} + s_{k-p_k} \text{ (since S is a decreasing sequence)} \\ &\leqslant t_k. \end{aligned}$$

Thus T is a decreasing sequence. *The sequence T obtained thus, is defined to be the product of R and S, and written R×S.*

The product of any number of sequences is easily seen to be commutative and associative.

The number of terms in a product sequence is easily shewn to be the sum of the numbers of terms in the factor sequences.

An important relation between the compound and the product is given by the following theorem :—

THEOREM II. *The Integral of the compound of any number of sequences is the product of their integrals.* In symbols:

$$\Delta^{-1}(R \oplus S \oplus T \oplus \dots) = \Delta^{-1} R \times \Delta^{-1} S \times \Delta^{-1} T \times \dots$$

It will be sufficient to indicate the proof for two sequences $R = (r_0\ r_1 \dots r_n)$, $S = (s_0\ s_1 \dots s_m)$.

Clearly, the $(k+1)$th term of the integral of their compound is :

$\Sigma r + \Sigma s$ — the sum of the k greatest numbers of the set of the r's and the s's.

But, the sum of the k greatest numbers of the set of the r's and the s's is the greatest of the quantities which are the sums of the λ greatest r's and the $k-\lambda$ greatest s's $(\lambda = 0, 1 \dots k)$. Thus, the $(k+1)$th term of the integral of the compound is the *least* of the quantities of the type :

(Σr — the sum of the λ greatest r's) + (Σs — the sum of the $k-\lambda$ greatest s's) $(\lambda = 0, 1 \dots k)$.

But this is precisely the $(k+1)$th term of the product of the integrals of the sequences R and S.

The theorem is thus proved.

As a special case, we have the result that the product of the one-term sequences (a), (b), (c) ..., where $a \geqslant b \geqslant c \cdots$, is the integral of the sequence $(a, b, c, \ldots)$.

Ex. Investigate the condition that a sequence R may be divisible by a sequence S.

§ 6 The Weierstrassian Theory of the Matrix.*

If M be a given matrix, the values of λ for which the determinant

$$| M - \lambda |$$

is equal to zero are the *characteristic* or the *latent roots* of M. If α is an r-ple characteristic root of M, then the determinant $| M - \lambda |$ would be divisible by $(\lambda - \alpha)^r$; it will happen further that all the kth minors of this determinant would be also divisible by a finite power of $\lambda - \alpha$, say $(\lambda - \alpha)^{r_k}$ $(k = 1, 2 \ldots)$. It may be shown that the sequence

$$R = (r_0 \, r_1 \, r_2 \ldots), \quad (r_0 = r)$$

(which may be called the *factor-sequence*) is not only a decreasing sequence, but a *differentiable* sequence; that is to say, the sequence

$$S = (s_0 \, s_1 \ldots) \quad (s_0 = r_0 - r_1, \cdots s_k = r_k - r_{k+1}),$$

(which will be called the *invariant-sequence*) is also a decreasing sequence, and in fact $(\lambda - \alpha)^{s_k}$ $(k = 0, 1 \ldots)$ are the invariant factors of M with respect to the characteristic root α.

The essence of Weierstrass' method consists in showing that the factor-sequence R, or what comes to the same thing, the invariant sequence S, completely characterises the relation of M to its multiple characteristic root α; or, in other words, *the necessary and sufficient condition that it may be possible to transform two matrices M, M′ into one another, by a relation of the form*

$$M' = \Delta M \Delta' \; (|\Delta| \neq 0, |\Delta'| \neq 0),$$

is the equality of their invariant-factors.

If M, M′ are interpreted as bilinear forms in two sets of variables, it would follow that the equality of the invariant factors is the condition for the transformability of the two forms into one another, by independent linear transformations of the two sets of variables. If M, M′ are considered as collineations between two spaces Σ, Σ', the equality of their invariant-factors gives the condition that M, M′ represent the *same* collineation under different co-ordinate systems.

---

* For an exposition of this theory, see the Cambridge Tract on *Quadratic Forms* or *Encylopédie des Sciences Mathematiques*, Tome I (ii) Fas. 4. or Hilton's *Linear Substitutions*.

§ 7. Compartite Matrices.*

A matrix M is said to be a *compartite* matrix, if it is of the form:

$$M = \left|\begin{array}{c|c} \Delta' & \\ \hline & \Delta'' \end{array}\right|,$$

where Δ', Δ'' are two matrices, and the blank spaces are occupied by zero elements; Δ', Δ'' are the *parts* of M. A compartite matrix with more than two parts is similarly defined. Suppose now that the factor-sequences of a latent root α with respect to the matrices Δ', Δ'' and M are respectively

$$R' = (r_0' \, r_1' \, r_2' \ldots)$$
$$R'' = (r_0'' \, r_1'' \, r_2'' \ldots)$$
$$R = (r_0 \, r_1 \, r_2 \ldots).$$

To obtain the sequence R from R', R'', we notice by inspection that the minor of any zero term in the determinant $|M - \lambda|$ vanishes. Hence it follows that any non-vanishing kth minor of $|M - \lambda|$, is a compartite determinant, the two parts of which are respectively, a μth minor of $|\Delta' - \lambda|$, and a $(k - \mu)$th minor of $|\Delta'' - \lambda|$ $(\mu = 0, 1, 2 \ldots k)$. Since $(\lambda - \alpha)^{r_k}$ is the highest power of $(\lambda - \alpha)$ which divides all the kth minors of $|M - \lambda|$, it follows from this that r_k is equal to the least of the numbers $(r'_\mu + r''_{k-\mu})$ $(\mu = 0, 1 \ldots k)$; that is to say, the sequence R is the *product* of the sequences R' and R''.

Thus we have:

> *The factor-sequence (with respect to a particular latent root) of a compartite matrix is the product of the factor-sequences of its parts.*

* The term 'compartite matrix' (due to Dr. Cullis *Matrices and Determinoids*, Vol. 2) suggests that the matrix is built up all in *compartments*. The usual name for such matrices is 'the direct product'—which I believe is one of the worst specimens of bad naming in mathematics. The same name occurs in other branches—as the *direct product* of two linear algebras, and the *direct product* of two groups. In both cases, I think it might be advantageously replaced by 'compartite.'

The art of right naming appears to depend on a happy intuition of the essential feature of the object to be named. Sylvester is probably one of the best *namers* in mathematics. His names besides being always appropriate, are redolent of a lively picturesqueness. One could not but wish there were more of such names in mathematics.

§ 8. **The Geometrical Theory of the Matrix.***

Let the matrix $M = | m_{rs} |$ of order n, be taken to represent the collineation

$$x'_k = m_{k1} x_1 + m_{k2} x_2 \cdots + m_{kn} x_n \quad (k = 1, 2 \ldots n)$$

in a space S_n of order n (or $n - 1$ dimensions). Then to search for the points which are transformed into themselves—(the 'latent' points of the collineation), we put $x'_k = \lambda x_k$, and eliminating the x's we obtain the *characteristic* equation for λ:

$$| M - \lambda | = 0.$$

If the roots of this equation, that is, the *latent* roots of the matrix be $\lambda_1 \lambda_2 \cdots \lambda_n$, then to each latent root λ_k corresponds in generally a definite latent point P_k, and *vice versa*. Suppose however that all† the n latent roots become equal to α. Then all the latent points P would coalesce and the actual manner of their coalescence could be investigated by examining what happens if all the latent roots are *nearly* equal to α. It would then appear that

(1) A certain number u_0 of the points P, say $P_1 P_2 \cdots P_{u_0}$ lie in a *definite* region Q_{u_0} of order u_0, in the limit. Every point of Q_{u_0} is a latent point of the matrix, and Q_{u_0} itself is called the 'latent' region.

(2) The latent region Q_{u_0} together with a certain number t_1 of the remaining points P lies in a *definite* region Q_{u_1} of order $u_1 = u_0 + t_1$, in the limit. Q_{u_1} is called the semilatent region of the first species.

(3) Q_{u_1} together with a certain number t_2 of the remaining points P lies in a definite region Q_{u_2} (the semi-latent region of the second species) of order $u_2 = u_1 + t_2$, in the limit; and so on.

The sequence $U = (u_0 u_1 \ldots)$, which we may call *the sequence of latent orders*, completely characterises the behaviour of the matrix in regard to its multiple latent root. This sequence being an increasing one, we use instead of it the *latency-sequence* T defined by

$$T = (t_0 \, t_1 \, t_2 \cdots); \; t_0 = u_0, \, t_1 = u_1 - u_0 \, ; \; t_k = u_k - u_{k-1}.$$

From the property of the matrix, it may be shewn that the latency sequence T is a decreasing one. (*Cf.* Whitehead, *l.c.*)

---

* For an exposition of this theory, see Whitehead's *Universal Algebra*.

† There is no loss of generality in this supposition, for a matrix with several distinct latent roots can be expressed as a compartite matrix, each part of which has only one latent root.

Further by reducing the matrix to a canonical form (as in Hilton : *Linear Substitutions*), it can be shewn that *the latency sequence corresponding to any latent root is the conjugate sequence of the invariant sequence for the same latent root.* This relation between the two sequences is a fundamental one.

§ 9. Geometrical Theory of Compartite Matrices.

Compartite matrices admit of a simple and natural interpretation in the geometrical theory. Let P_r, P_{n-r} be two non-intersecting regions of orders r and $n-r$ in a space S_n of order n, so that S_n is the *join* of P_r and P_{n-r}. Through any point of S_n there can be drawn one and only one straight line to meet P_r and P_{n-r}; hence any two collineations Δ', Δ'' in the spaces P , P_{n-r} respectively determine a unique collineation M of S_n. If the n points of reference for homogeneous co-ordinates in S_n are so chosen that r of them are points of reference in P_r and the remaining $n-r$ are the points of reference in P_{n-r}, then it will be immediately evident that the *matrix* M is the compartite matrix whose parts are the *matrices* Δ' and Δ''. Conversely, we may speak of the collineation Δ' as the P_r - component of the collineation M.

From this geometrical relation between the collineations M, Δ' and Δ'', it is immediately inferred that the semi-latent region of the kth species* of M is the join of the semi-latent regions of the kth species of Δ' and Δ''. Hence, the sequence of latent orders of a compartite matrix (with reference to a particular latent root) is the sum of the sequences of latent orders of its parts; and therefore it follows that *the latency sequence of a compartite matrix is the sum of the latency sequences of its parts.*

§ 10. Relations between the various sequences.

Suppose first that the algebraic view-point is fundamental. Then, from what has been proved (§ 7), the factor-sequence of a compartite matrix is the *product* of the factor-sequences of its parts. But the

---

* If $l<k$ and the matrix Δ' has only l species of semi-latent regions, then for the validity of this theorem, the semi-latent region of the kth species of Δ' should be regarded as identical with that of the highest or the lth species.

factor-sequence is the integral of the invariant sequence. Hence by Theorem II, we have :

*The invariant sequence of a compartite matrix is the compound of the invariant sequences of its parts** ... ... (A)

Assuming now that the latency sequence is the conjugate of the invariant-sequence—we have, on using Theorem I, the theorem proved in § 9, namely,

The latency-sequence of a compartite matrix is the sum of the latency sequences of its parts... ... ... (B).

If on the other hand, the geometrical theory is considered fundamental, we could deduce (using the fact that the latency and invariant sequences are conjugates of one another), by using Theorems I and II, the theorem (A) and the theorem about the factor-sequence from (B).

---

* This theorem is so fundamental in matrix-theory that it ought to find a place in all text-books on the subject. Hilton in his *Linear Substitutions* comes very near stating it, but changes his mind apparently at the last moment and passes on, leaving the reader in doubt. For a discussion of the theorem, see page 458 of the *Encyclopedie* already quoted.

Twin Points.

By M. Bhimasena Rao.

1. *Introduction.*

*Twin points** with respect to a triangle ABC, are diametrically opposite points on a rectangular hyperbola circumscribing the triangle. Since H, the orthocentre of the triangle, lies on every circumscribing rectangular hyperbola, points which are twins with respect to the triangle ABC are also twins with respect to the triangles HBC, HCA, HAB. Since the triangles formed by the points A, B, C, H, have a common nine-point centre, twins which are collinear with the nine-point centre have a symmetrical relation with respect to the four triangles of this orthocentric system.† In this paper are considered certain interesting geometrical properties of such twins and their isogonal conjugates with respect to the four triangles of the orthocentric system.

2. Theorem I.

If twins P, Q, with respect to a triangle ABC are collinear with N, the nine-point centre, and if P_1 is the isogonal conjugate of P, then PP_1 is parallel to HQ.

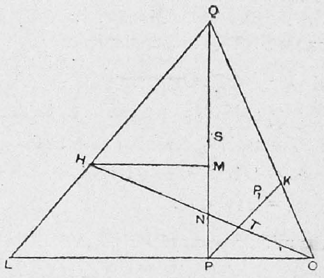


Fig 1.

* W. Gallatly in his *Modern Geometry of the Triangle*, (second edition), pp. 62–64, defines twins as isogonal conjugates of a pair of inverse points, and mentions the interesting relation that twins have similar anti-pedal triangles, the difference of whose areas is four times the area of the triangle of reference.

† Such twins presented themselves in the study of *Morley's Theorem* published in the *Journal of the Indian Mathematical Society* for April 1920.

Let O be the circum-centre of ABC. Join OP and produce it to meet HQ in L; on PQ take NM equal to PN, and join HM. Let PP_1 meet OQ in K, and let S, T be the middle points of PQ, PP_1 respectively.

Since S, the centre of the rectangular hyperbola ABCPQ, lies on the nine-point circle,

$$\begin{aligned} R &= 2\,NS = 2\,(NM + MS) \\ &= PM + 2\,MS \\ &= PS + MS \\ &= QS + MS \\ &= QM. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1) \end{aligned}$$

Since P and P_1 are isogonal conjugate points, we have by a theorem of Mr. V. Ramaswamy Aiyar,*

$$\begin{aligned} OP\,.\,OP_1 &= 2R\,.\,NT \\ &= 2QM\,.\,NT, \text{ by (1)}. \qquad \ldots \qquad \ldots \qquad (2) \end{aligned}$$

The pedal circle of P with respect to ABC passes through S. The radical axis of the pedal circle and the nine-point circle is perpendicular to NT, the join of their centres, and cuts the nine-point circle at an angle equal to SNT or its supplement. Hence by a result given by the author† angle POP_1 = angle SNT or its supplement. ... ... (3)

$$\begin{aligned} \therefore\ \Delta\,(POP_1) &= \tfrac{1}{2}\,OP\,.\,OP_1\,.\,\sin POP_1 \\ &= QM\,.\,NT\,.\,\sin SNT, \text{ by (2) and (3)} \\ &= (QN - NM)\,NT\,.\,\sin SNT \\ &= 2\,(QNT) - 2\,(MNT) \\ &= 2\,(QMT). \end{aligned}$$

Since T is the middle point of PP_1,

$$\Delta\,(POP_1) = 2\,\Delta\,(POT)$$

$$\therefore \qquad \Delta\,(POT) = \Delta\,(QMT) \qquad \ldots \qquad \ldots \qquad (4)$$

$$\text{Again } \frac{OK}{KQ} = \frac{\Delta(POT)}{\Delta(PTQ)} = \frac{\Delta(QMT)}{\Delta\,(PTQ)} = \frac{QM}{QP}, \text{ by (4)} \qquad \ldots \qquad (5)$$

---

* *Vide* W. Gallatly : *Modern Geometry of the Triangle*, p. 79, 2nd Edition.

† *Vide J. I. M. S.*, Question 876.

Since ON = NH, and PN = NM, HM is equal and parallel to OP.

$$\therefore \quad \frac{QM}{QP} = \frac{HM}{PL} = \frac{OP}{PL}.$$

$$\therefore \quad \frac{OK}{KQ} = \frac{QM}{QP} = \frac{OP}{PL}, \text{ by } (5).$$

Hence PK is parallel to QL. That is, PP_1 is parallel to QH.

3. THEOREM II.

If Q_1 is the isogonal conjugate of Q in Theorem I, then PP_1 and QQ_1 intersect on the circum-circle of ABC.

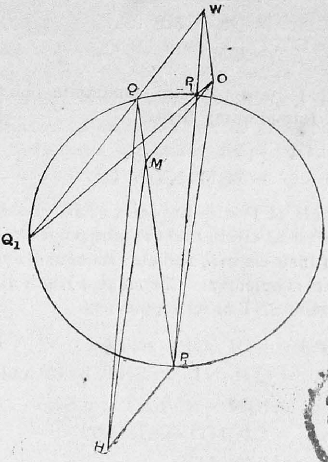


Fig. 2.

185293

Let W be the point of intersection of PP_1 and QQ_1.

By Theorem I,

HP is parallel to QQ_1,

and HQ is parallel to PP_1

$\therefore$ HQWP is a parallelogram.

Since PM and OH bisect each other at N,

HMOP is a parallelogram.

$\therefore$ MQWO is a parallelogram. ... ... (6)

Hence OW is equal and parallel to QM which has been proved in (1) to be equal to the radius of the circle ABC. Therefore the joins of isogonal pairs (PP_1) and (QQ_1) intersect on the circumcircle of ABC.

4. THEOREM III.

Twins collinear with the nine-point centre are concyclic with their isogonal conjugates.

With the previous notation, since P_1, Q_1 are isogonal conjugates of twin points, they are inverse points with respect to the circle ABC.*

$$\therefore\ OP_1 \cdot OQ_1 = R^2 = OW^2, \text{ by Theorem II}$$
$$\therefore\ \text{angle } OWP = \text{angle } OQ_1W.$$

Since OW is parallel to PQ, by (6),

$$\text{angle } OWP = \text{angle } QPP_1$$
$$\therefore\ \text{angle } P_1Q_1Q = \text{angle } QPP_1.$$

Hence P, Q, P_1, Q_1 are concyclic, which proves the theorem.

The converse results corresponding to Theorems II and III are true, but cannot be proved by retracing the steps. We shall give a separate demonstration using an interesting theorem in the theory of similar figures.

5. THEOREM IV.

If twins are concyclic with their isogonal conjugates, then the joins of isogonal pairs intersect on the circum-circle of the triangle of reference.

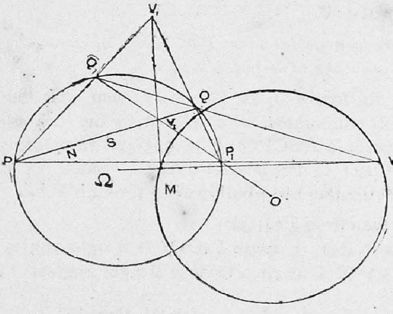


Fig. 3.

* *Vide* Gallatly : *Modern Geometry of the Triangle,* p. 62.

In the figure, P, Q are the twins; P_1, Q_1 their isogonal conjugates, with respect to a triangle ABC, and W the point of intersection of the joins of isogonal pairs PP_1, QQ_1; V and V_1 are the meets of PQ_1, QP_1, and PQ, P_1Q_1 respectively. The centre of the circle PQ_1QP_1 is Ω, and ΩW cuts VV_1 at M.

Now, VV_1W is the diagonal triangle of PQ_1QP_1.

$\therefore$ VV_1 is the polar of W with respect to the circle (Ω), and M is the inverse of W with respect to the same circle.

Next, consider directly similar figures described on the chords PQ and Q_1P_1. Their double point is the inverse, with respect to the circle (Ω), of the meet of the joins of the non-corresponding pairs PP_1 and QQ_1.*

Hence M is the double point. ... ... (7)

Again, P,P_1 and Q,Q_1 are isogonal pairs with respect to the triangle ABC. The double point of directly similar figures described on PQ and Q_1P_1 lies, by a theorem of Mr. V. Ramaswamy Aiyar, on the circum-circle of ABC.† Therefore the double point M lies on the circle ABC. (8)

Since P_1, Q_1 are the isogonal conjugates of P, Q, they are inverse points with respect to the circle ABC. The circle PQ_1QP_1 passing through these inverse points is orthogonal to the circle ABC. Since M lies on the circle ABC by (8), W which is the inverse point of M with respect to the circle PQ_1QP_1 lies on the circle ABC, and thus the theorem is proved.

6. THEOREM V.

If twins are concyclic with their isogonal conjugates, then their join passes through the nine-point centre.

Referring to figure 3, it is easily seen that the points V,V_1 are also isogonal conjugates with respect to the triangle ABC.‡ The rectangular hyperbola ABCPQ is the isogonal transformation of the line OP_1Q_1 with respect to the triangle ABC. Since V_1 lies on the line OP_1Q_1, the rectangular hyperbola passes through V. ... ... (9)

Again, in the circle PQ_1QP_1,

$$\text{angle } P_1PQ_1 + \text{angle } P_1QQ_1 = 2 \text{ right angles};$$

i.e. angle VPW + angle VQW = 2 right angles.

---

* *Vide. J.I.M.S.*, Vol. IV, Page 175.

† *Vide J.I.M.S.*, Vol. II, Question 125.

‡ *Vide J.I.M.S.*, Vol. IV, Page 173.

Hence V being a point on the rectangular hyperbola ABCPQ by (9), and the line VW subtending supplementary angles at the diametrically opposite points P and Q of this hyperbola, it follows that the hyperbola passes also through W. ... ... (10)

Since P_1, Q_1 are inverse points with respect to the circle ABC,

$$OP_1 . OQ_1 = R^2 = OW^2, \text{ by Theorem IV.}$$

$\therefore$ angle OWP_1 = angle OQ_1W = angle QPW,
in the cyclic quadrilateral QQ_1PP_1.

Hence OW is parallel to PQ. ... ... (11)

Now, the fourth point of intersection of the rectangular hyperbola ABCPQ and the circle ABC is W. If S is the centre of the rectangular hyperbola, we know that the point H on WS produced such that HS is equal to SW is the orthocentre of the triangle ABC.

In the triangle OWH, S is the middle point of WH, and the line QSP, being parallel to OW by (11), bisects OH. Hence PQ passes through the nine-point centre.

7. THEOREM VI.

If P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 are respectively the isogonal conjugates of the twins P, Q, (collinear with the nine-point centre), with respect to the four triangles of the ortho-centric system, then the points P, Q, P_1, P_2, P_3, P_4 and P, Q, Q_1, Q_2, Q_3, Q_4 lie on two rectangular hyperbolas concentric with the rectangular hyperbola PQABCH.

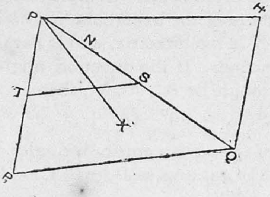


Fig. 4.

Draw the tangent PX to the rectangular hyperbola ABCPQ. Let P_1 be the isogonal conjugate of P with respect to the triangle ABC, and S, T the middle points of PQ, PP_1 respectively.

Since PQ is a diameter of the rectangular hyperbola and H is a point on it,

$$\text{angle } HPX = \text{angle } HQP.$$

Because HQ and PP_1 are parallel, by Theorem I,

$$\text{angle } HQP = \text{angle } P_1PQ.$$

$$\therefore \quad \text{angle } HPX = \text{angle } P_1PQ. \qquad \ldots \quad (12)$$

The pedal circle of P cuts the nine-point circle at S. The angle of intersection of the two circles is, by a theorem of Mr. V. Ramaswamy Aiyar,* equal to

$$90^\circ - (PAB + PBC + PCA).$$

Hence $\quad NST = 90^\circ - (PAB + PBC + PCA). \qquad \ldots \quad (13)$

If the direction angles of the tangent PX, referred to the triangle ABC be α, β, γ then

$$\begin{aligned} \text{angle } PAB &= \text{angle } APX - \alpha \\ ,, \quad PBC &= \quad ,, \quad BPX - \beta \\ ,, \quad PCA &= \quad ,, \quad CPX - \gamma. \end{aligned}$$

$$\therefore \quad NST = 90 + \alpha + \beta + \gamma - (APX + BPX + CPX).$$

Since TS is parallel to P_1Q,

$$\text{angle } NST = \text{angle } P_1QP.$$

From (12) and (13), we have

$$P_1PQ - P_1QP = HPX + APX + BPX + CPX - (90^\circ + \alpha + \beta + \gamma). \quad (14)$$

Next, consider the reciprocal of the nine-point circle with respect to the rectangular hyperbola ABCPQ. Since the hyperbola is self-conjugate with respect to the ortho-centric triangle, the reciprocal is a parabola, having S, the centre of the hyperbola, for its focus. The polar of the nine-point centre N is the directrix, and is parallel to the tangent PX, since S, N, P are in a line. If the direction angles of PX with respect to the ortho-centric triangle be α', β', γ', then

$$\text{angle } SPX = 90^\circ - (\alpha' + \beta' + \gamma').\dagger \qquad \ldots \quad (15)$$

Since the sides of the orthocentric triangle cut the corresponding sides of the triangle ABC at angles B − C, C − A, A − B, we have,

$$90 - (\alpha' + \beta' + \gamma') = 90 - (\alpha + \beta + \gamma).$$

Hence $\quad QPX = SPX = 90 - (\alpha + \beta + \gamma). \qquad \ldots \quad (16)$

Substituting in (14), we get

$$P_1PQ - P_1QP = HPX + APX + BPX + CPX + QPX - 180^\circ. \quad (17)$$

---

* *Vide J.I.M.S.*, Vol. IX, Page 167.

† *Vide* Gallatly's *Modern Geometry of the Triangle*, § 41, Page 27.

The symmetry of the result (17) shows that the difference of the base angles of the triangles P_1PQ, P_2PQ, P_3PQ, P_4PQ, is the same. By a well known result, the points P_1, P_2, P_3, P_4 lie on a rectangular hyperbola having PQ as a diameter. Similarly Q_1, Q_2, Q_3, Q_4, lie on another rectangular hyperbola having PQ as a diameter.

8. THEOREM VII.

If the pedal circles of P with respect to the four triangles of the orthocentric system are drawn, then the inverses of P with respect to these circles are collinear with the centre of the rectangular hyperbola ABCPQ.

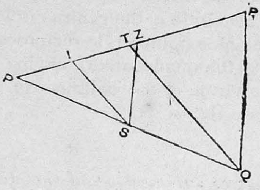


Fig. 5.

Let T be the centre of the pedal circle of P with respect to the triangle ABC, and I the inverse of P with respect to the pedal circle. Draw the tangent QZ to the rectangular hyperbola PQ $P_1P_2P_3P_4$. Join S, the centre of the rectangular hyperbola to T and I.

Since TS is the radius of the pedal circle of P,

$$TI.\ TP = TS^2.$$

$$\therefore \quad \text{angle } TSI = \text{angle } TPQ.$$

Since QZ is a tangent to the rectangular hyperbola PQP_1,

$$\text{angle } P_1QZ = \text{angle } P_1PQ.$$

$$\therefore \quad \text{angle } TSI = \text{angle } P_1QZ$$

But TS is parallel to P_1Q;

$$\therefore \quad SI \text{ is parallel to } QZ.$$

Hence I, the inverse of P, lies on the diameter conjugate to PQ in the rectangular hyperbola $P_1P_2P_3P_4$, and the result stated in the theorem easily follows.

When P is an arbitrary point, it may be shown that the inverses of P with respect to the pedal circles of P drawn as before, will lie on a circle passing through S, the centre of the rectangular hyperbola ABCP.

The present theorem shows that when P, S, N are collinear, the circle degenerates into a line.

9. Let us now reciprocate the result stated in the last theorem, with respect to a circle, centre P. The points A, B, C, H reciprocate into four lines, having P as a common point of intersection of the director circles of the conics touching these lines. The rectangular hyperbola ABCP reciprocates into a parabola touching these lines and having as focus the inverse of S,—in other words, the inverse of S is the Miquel point of the four lines. The line of collinearity of the four inverses of P referred to in the last theorem, inverts into a circle passing through P and the Miquel point. This circle is the centre circle of the quadrilateral formed by the lines. The nine-point circle reciprocates into an inconic of the diagonal triangle of the quadrilateral, having P as focus and the Miquel point as vertex in virtue of the collinearity of P, S, N. Hence we obtain the following result :—

THEOREM VIII.

P is a common point of intersection of the director circles of the system of conics touching the sides of a quadrilateral whose diagonal triangle is XYZ. If the conic inscribed in XYZ with P as focus has its vertex at the Miquel point of the quadrilateral, then the centre circle of the quadrilateral passes through P.

10. Next, consider the centre circle of the quadrilateral, referred to in Theorem VIII. It is the locus of the double foci of all circular cubics passing through the six points of intersection of the sides of the quadrilateral, because tangents at two of the common points of a pencil of cubics intersect on a conic passing through these two points. The ninth common point of the above pencil of circular cubics is the Miquel point. Under the conditions of Theorem VIII the point P lies on the centre circle. Hence with P as double focus, a circular cubic can be described passing through the six points of intersection of the sides of the quadrilateral and the Miquel point. It has been shown that the normal to the cubic at the Miquel point passes through the double focus P*. Since the inconic of the diagonal triangle having P as focus has its vertex at the Miquel point, the circular cubic touches the inconic and its auxillary circle at the Miquel point.

Reciprocating the cubic with respect to a circle centre P, it is easily seen that we obtain Morley's theorem, *viz.*

---

* *Vide*: The author's paper on *Morley's Theorem, J. I. M. S.*, April 1920.

Every curve of class three which touches the six joins of four orthocentric points and passes through the circular points touches the nine-point circle.

P being the double focus of the circular cubic is also the double focus of the curve in Morley's theorem. The above discussion shows that the point of contact of the nine-point circle and the curve lies on the line joining the nine-point centre and the double focus of the curve, and that it is the centre of the rectangular hyperbola passing through the orthocentric points and the double focus. The inverse of a circular cubic with respect to its double focus being a bi-circular quartic having the same double focus, the inverse theorem may be stated thus*:—

THEOREM IX.

If twin points P, Q, are collinear with the nine-point centre, a bi-circular quartic can be drawn with P as double focus passing through P and the feet of the perpendiculars from P on the six joins of the orthocentric points A, B, C, H; this quartic passes through the centre of the rectangular hyperbola ABCP, and touches the nine-point circle at that point.

We have also established incidentally

THEOREM X.

Twins collinear with the nine-point centre are the double foci of the curves in Morley's theorem.

11. We shall now conclude with some remarks regarding the equation to the locus of the double foci referred to in the last theorem.

Taking the orthocentric triangle DEF as a triangle of reference, the trilinear equation to the locus is easily seen to be

$$\begin{vmatrix} \dfrac{\cos D}{x} & \dfrac{\cos E}{y} & \dfrac{\cos F}{z} \\ \dfrac{\sin D}{y^2-z^2} & \dfrac{\sin E}{z^2-x^2} & \dfrac{\sin F}{x^2-y^2} \end{vmatrix} = 0;$$

$$\text{i.e.,} \quad \frac{\sin D\,.(y\cos F - z\cos E)}{y^2-z^2} + \text{two similar terms} = 0. \quad \dots \quad (18)$$

* If instead of the double focus, the centre of inversion is a single focus, the corresponding theorem has been stated by Mr. V. Ramaswami Aiyar, as follows:—

Given a triangle ABC, if a point P and the feet of the perpendiculars from it on the sides and altitudes of ABC all lie on a cartesian oval, then the negative pedal of the oval with respect to P touches the nine-point circle. (*Vide J.I.M.S.*, Volume 10, Question 960)

This is a curve* of the fifth degree circumscribing the orthocentric triangle and having double points at A,. B, C, H $(1, \pm 1 \pm 1)$. It passes through the nine-point centre (cos D, cos E, cos F). It is obvious geometrically that the nine-point centre and the sixteen in- and ex-centres of the four triangles of the orthocentric system lie on the curve, as in these cases the joins of the twins are easily seen to pass through the nine-point centre,—a fact noticed also by Mr. H. W. Richmond, F. R. S †

The curve is circular, and its remaining three points of intersection with the line of infinity lie on M'Cay's cubic ‡ of the orthocentric triangle. It touches the nine-point circle at the circular points and the remaining six points of intersection of the curve and the nine-point circle lie on another interesting cubic which is the locus of the centres of conics touching the sides of the orthocentric triangle at points where the normals are concurrent.§ In fact, equation (18) can be transformed into

$$C.S = I.Q. \quad \ldots \quad \ldots \quad (19)$$

where $C \equiv \cos \mathrm{D}\, x\,(y^2 - z^2) + \ldots\ldots + \ldots\ldots = 0$ is the M'Cay's cubic of DEF,

$S \equiv yz \sin \mathrm{D} + zx \sin \mathrm{E} + xy \sin \mathrm{F} = 0$ is the nine-point circle of ABC,

$I \equiv x \sin \mathrm{D} + y \sin \mathrm{E} + z \sin \mathrm{F} = 0$ is the line at infinity,

and $Q \equiv \cos \mathrm{D}\, yz\,(y^2 - z^2) + \ldots\ldots + \ldots\ldots = 0$,

is a circular quartic, which may be written as

$$\frac{S\,.\,R - I\,.\,M}{\lambda},$$

where $R \equiv x^2 (\sin^2 \mathrm{E} - \sin^2 \mathrm{F}) + \ldots\ldots + \ldots\ldots = 0$ is the rectangular hyperbola ABCN,

$M \equiv \sin \mathrm{E}\,.\, \sin \mathrm{F}\,.\, x\,(y^2 - z^2) + \ldots\ldots + \ldots\ldots = 0$ is the second cubic referred to above,

and λ is the constant $2 \sin \mathrm{D}\,.\, \sin \mathrm{E}\,.\, \sin \mathrm{F}$.

---

* The line at infinity forms part of the locus of the double foci inasmuch as any point on it may be regarded as the double focus of the three-cusped hypocycloid belonging to the system, and with the curve given in (18) constitutes the complete locus.

† *Vide* Edinburgh Mathematical Society's *Proceedings* for 1919—20, Vol. XXXVIII, pp. 2—5.

‡ The locus of isogonal conjugate points whose pedal circle touches the nine-point circle, *Vide* Gallatly's *Modern Geometry of the Triangle*, Second Edition, p. 80,

§ *Vide J. I. M. S.*, Question 430, wherein some properties of this cubic are stated.

Hence the equation to the curve may be written in the form,

$$\lambda\ CS - ISR + I^2M = 0. \quad \ldots \quad \ldots \quad (20)$$

An interesting form of the equation is obtained by the use of covariants of two conics S, S′ and the line $lx + my + nz$.

This form is

$$S^2L' + S'^2L = 2SS'\frac{L\Sigma' + L'\Sigma}{\phi} \quad \ldots \quad (21)$$

where S,S' are any two rectangular hyperbolas through A, B, C, H, in point co-ordinates, and Σ, Σ', their expressions in line co-ordinates,

$lx + my + nz$ is the line at infinity,

J is the Jacobian of S,S' and $lx + my + nz$,

L is the line joining the poles of $lx+my+nz$ with respect to J and S,

L' is the corresponding line with respect to S′,

and ϕ is the harmonic envelope of S and S'.*

If we take S and S' as the rectangular hyperbolas touching the line at infinity, then Σ and Σ' will vanish, L and L' will become the isotropic lines through the nine-point centre, and the equation (21) simply reduces to

$$S^2L' + S'^2L = 0, \quad \ldots \quad \ldots \quad (22)$$

putting in evidence the fact that the nine-point centre lies on the curve and that it is the double focus of the curve.

---

* *Vide* Salmon's *Conic Sections*. (1904), pages 344 and 363.

The result (21) is easily verified by taking

$$S = z^2 - x^2,\ S' = x^2 - y^2,$$

$$l = \sin D,\ m = \sin E,\ n = \sin F.$$

Whence $\Sigma = -\sin^2 E,\ \Sigma' = -\sin^2 F,$

$$\phi = -\sin^2 D + \sin^2 E + \sin^2 F,$$

and $L = \sin E\,(z\cos D - x\cos F)$

$$L' = \sin F\,(x\cos E - y\cos D).$$

Definition of Straight Line *

By S. V. Ramamurthy, M.A., I.C.S.

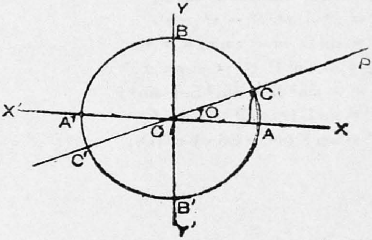
1. In his excellent little book "*Euclid's Parallel Postulate*" (The Open Court Publishing Company, 1905), Withers has stated that "the essential meaning of the straight line as used in the various forms of geometry has not been clearly determined." Lobatchewsky defines the straight line "as one which fits upon itself in all its positions so that, if we turn the surface containing it about two points of the line, the line does not move." This may be as correct a definition as that of man as a featherless biped ; but each gives equally little information about the essential nature of straight line or man. Euclid defines a straight line as the shortest distance between two points. But to define distance we need a straight line. Some are content to say that a straight line has the same direction throughout its length. But then direction is left undefined. We also say that a straight line has two directions.

There is thus no satisfactory definition of a straight line at present.

To analyse the nature of a straight line, take a straight line XOX′ and another perpendicular to it, YOY′, as the axes of x, y.

Describe a circle of unit radius cutting these axes at A,A′ ; B,B′. Let a line OP cut the circle at C, C′.

Draw the co-ordinates of C.



* Presented to the Fourth Conference of the Indian Mathematical Society, Poona, April 1924.

Let $\angle XOP = \theta$, $OA = 1$, $OB = i$, $OC = x + iy = e^{i\theta}$, $OA' = -1$, $OB' = -i$, $OC' = -e^{i\theta}$.

Along OX, mark off unit lengths AA_1, A_1A_2 ...... So too along OY unit lengths BB_1, B_1B_2 ...; and along OP unit lengths CC_1, C_1C_2, ...

Let OA now represent an atom of time. Then OX is a line of time particles and the lines OA, OA_1, OA_2, ... can be represented by the natural numbers 1, 2, 3 ......

So too, the lines OB, OB_1, OB_2, can be represented by i, $2i$, $3i$,...... If here, we take i itself as the unit, the lines are represented by the natural numbers 1, 2, 3, ...

So too, the lines OC, OC_1, OC_2 ...... can be represented by $e^{i\theta}$, $2e^{i\theta}$, $3e^{i\theta}$ ...... or taking $e^{i\theta}$ as unit, by the natural numbers 1, 2, 3 ......

Now, in my paper on the 'Definition of Number,' I have defined a natural number as a piece of time. Hence any part of the line OX is a piece of time. Hence OX is a line of time. So too OY and OP are lines of time.

No curved line through O can have its parts so represented by natural numbers. For, if we mark off unit lengths on it, $e^{i\theta}$ (which may be called the *density* of the length in relation to the plane) alters at each part of the curved line.

Thus we can define a straight line as a line of time.

This definition applies to any kind of continuum — of time, space, matter or mind. For, in all such continua there are time particles. A line of time particles gives a straight line.

In a space-time continuum of two dimensions, the space particles are all equal and a line of them has its parts represented by natural numbers and yields what may be regarded as a line of time. Hence, we have two straight lines determined by the space particles and the time particles. The theory of dimensions based on the presence of independent sets of equal particles has already been described in my paper on "Time, Space, Matter and Mind." The view of straight line presented here is consistent with that theory of dimensions and supports it.

JOURNAL

OF

THE INDIAN MATHEMATICAL SOCIETY

Volume XV, No. 9.

June 1924.

NOTES AND QUESTIONS.

Notes and Questions.

Note on the representation of a Number as the sum of two others satisfying given conditions.

1. The object of this note is to discuss some of the properties of the triads of numbers of the types (written in the usual arithmetical notation)

$$a_1\, a_2\, a_3 \,\cdots\cdots\, a_p,$$
$$b_1\, b_2\, b_3 \,\cdots\cdots\, b_p,$$
$$c_1\, c_2\, c_3 \,\cdots\cdots\, c_p,$$

in γ-ary scale $[\gamma = (3p+1)]$, where no two of the a's, b's, c's are equal and none of them is zero and the sum of the first two numbers is equal to the third number.

It is, at once, obvious that all the numbers from 1 to $3p$ are distributed among the digits of the three numbers and that $c_1 \not< 3$.

$$\text{Evidently } \Sigma a_\gamma + \Sigma b_\gamma + \Sigma c_\gamma = 1 + 2 + 3 + \cdots\cdots + (\gamma - 1)$$
$$= \frac{\gamma(\gamma-1)}{2}; \quad \cdots \quad \cdots \quad (1)$$

and
$$\gamma^{p-1}(a_1 + b_1 - c_1) + \gamma^{p-2}(a_2 + b_2 - c_2) + \cdots\cdots$$
$$+ \gamma^{p-s}(a_s + b_s - c_s) + \cdots\cdots (a_p + b_p - c_p) = 0.$$

Now the successive values of $a_s + b_s - c_s$, $a_{s-1} + b_{s-1} - c_{s-1}, \cdots\cdots$ admit of one or more of the following types of sequences :—

(i) zeroes

(ii) $\gamma, -1$

(iii) $\gamma, \gamma - 1, -1$

(iv) $\gamma, \gamma - 1, -\gamma - 1,$

etc.

Suppose the p numbers $a_1 + b_1 - c_1$, $a_2 + b_2 - c_2$, $\cdots\cdots$ $a_p + b_p - c_p$ contain x zeroes, y_0 groups of the type (ii), y_1 groups of the type (iii), y_2 groups of the type (iv), and so on.

$$\text{Then} \quad \Sigma a_\gamma + \Sigma b_\gamma - \Sigma c_\gamma = (\gamma - 1)(y_0 + 2y_1 + 3y_2 + \cdots\cdots)$$
$$= (\gamma - 1)k, \quad \text{(say)};$$

i. e., $$y_0 + 2y_1 + 3y_2 + \dots\dots = k = \frac{\Sigma a_\gamma + \Sigma b_\gamma - \Sigma c_\gamma}{\gamma - 1}. \quad \dots \quad (2)$$

Also, $$x + 2y_0 + 3y_1 + 4y_2 + \dots\dots = p. \quad \dots \quad \dots \quad (3)$$

From (2) and (3),

$$x + y_0 + y_1 + y_2 + \dots\dots = p - k, \quad \dots \quad (4)$$

which shows that k may take all values from 0 to $p-1$.

From (1) and (2),

$$\Sigma c_\gamma = \frac{(\gamma - 1)(\gamma - 2k)}{4}, \quad \dots \quad \dots \quad (5)$$

which is one of the necessary conditions that $[c_1\, c_2 \dots\dots c_p]$ may admit of the representation in question.

The upper and lower limits of Σc_γ are obtained by putting $k = 0$ and $(p-1)$ respectively in (5);

i.e., $$\frac{(\gamma - 1)(\gamma + 8)}{12} < \Sigma c_\gamma < \frac{\gamma(\gamma - 1)}{4}. \quad \dots \quad (6)$$

Hence,

(i) when $\gamma = 12m + 1$, k can take all integral values from 0 to $4m - 1$,

(ii) $\gamma = 12m + 4$, k must be even, for otherwise $(\gamma - 1)(\gamma - 2k)$ will not be divisible by 4; and hence k can take all even values from 0 to $4m$;

(iii) when $\gamma = 12m + 10$, k must be odd, for the same reason as before and therefore can take all odd values from 1 to $4m+1$;

(iv) when $\gamma = 12m + 7$, $(\gamma - 1)(\gamma - 2k)$ is not divisible by 4 and hence the representation in question is impossible.

2. When the number $(c_1\, c_2 \dots\dots c_p)$ admits of the representation in question,

$$\frac{(p-k)!}{x!\; y_0!\; y_1! \dots\dots}$$

permutations of the number also admit of a similar representation. For, we may so permute the digits $c_1, c_2, \cdots c_p$ that the numbers of each of the groups of digits corresponding to the types (i), (ii), (iii) ... of § 1, occur together though the relative positions of the different groups may be altered.

3. It is interesting to consider the particular case $\gamma = 10$ in some detail.

Equations (2), (3) and (5) above reduce in this case to

$$y_0 + 2y_1 = k\,;\ x + 2y_0 + 3y_1 = 3\,;\ \text{and } c_1 + c_2 + c_3 = \frac{9\,(5-k)}{2}.$$

Consequently, k cannot take the values 0 and 2. Since $k<3$, $k=1$.

Hence the only possible case is when

$$y_0 = 1,\ y_1 = 0,\ x = 1\,;$$

and

$$c_1 + c_2 + c_3 = 18.$$

Thus if $a_1\ a_2\ a_3 + b_1\ b_2\ b_3 = c_1\ c_2\ c_3$, the digits should satisfy either of the following sets of equations:—

$$\text{(I)}\quad \left.\begin{aligned} c_1 + c_2 + c_3 &= 18 \\ a_1 + b_1 - c_1 &= -1 \\ a_2 + b_2 - c_2 &= 10 \\ a_3 + b_3 - c_3 &= 0. \end{aligned}\right\} \qquad \text{(II)}\quad \left.\begin{aligned} c_1 + c_2 + c_3 &= 18 \\ a_1 + b_1 - c_1 &= 0 \\ a_2 + b_2 - c_2 &= -1 \\ a_3 + b_3 - c_3 &= 10. \end{aligned}\right\}$$

The four equations in (I) or (II) are not all independent; for, any one of them can be deduced from the other three. Thus it is sufficient for us to consider three equations only in (I) and (II).

We shall denote that representation of $c_1\, c_2 \cdot c_3$ in which $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$ by the term *primitive* representation. From each such primitive representation, we can get three more by permuting between a_1, b_1; a_2, b_2; and a_3, b_3.

It can be shown by examining the behaviour of the 2-partitions of $(c_1 - 1)$ and c_3 or of c_1 and $(c_2 - 1)$ that for given values of $c_1\, c_2\, c_3$ satisfying the condition $c_1 + c_2 + c_3 = 18$, there are at most two primary solutions of (I) or (II), while

neither may have a solution, as when $c_1 = 7,\ c_2 = 6,\ c_3 = 5$;
or (I) alone may have no solution, as when $c_1 = 9,\ c_2 = 8,\ c_3 = 1$;
or (II) .... $c_1 = 9,\ c_2 = 3,\ c_3 = 6$.

In no case does the total number of primitive representations of any number of the form $(c_1 c_2 c_3)$, viz., $[100c_1 + 10c_2 + 18 - (c_1 + c_2)]$ exceed two.

The number of solutions of (I) and (II) for all numbers of three digits whose sum is 18 satisfying our conditions is given below in a tabular form:—

Numbers.	No. of primitive solutions of	
	(I)	(II)
918; 837; 819; 783; 675.	2	0
945; 864.	1	1
936; 927; 738; 729; 648; 639; 576; 549; 468; 459.	1	0
486; 972; 963; 954; 873; 495; 846; 792; 693; 657; 594.	0	1
981; 891; 567.	0	2
765; 756; 684; 396; 387; 378; 369; 297; 279; 198; 189.	0	0

NOTE.—As an exercise on the application of partitions of the type discussed here, one may investigate the arrangement of the digits 0, 1, 2, 3, ... 9 in such a way as to form a number divisible by all the numbers from 1 to 17. It is found that there are only *four* such arrangements, *viz*:

(i) 2438195760
(ii) 3785942160
(iii) 4876391520
(iv) 4753869120.

A. A. KRISHNASWAMI AYYANGAR.

A Note.

It may be of interest to note that the value of 2^{100} is

1267650600228229401496703205376.

Persons interested may attempt at factorising $(2^{100} + 1)$.

HANS R. GUPTA.

Solutions.

Question 1229.

(Prof. K. J. SANJANA) :—Show how to find the equation of the surface traced out by the shortest distances of all points of a given straight line from a given conic section whose plane does not contain the straight line, and obtain the result when the conic is a circle.

Solution by K. Satyanarayana and K. Srinivasa Raghavan.

Case (i) Central Conic : Referred to its principal axes, let its equation be

$$ax^2 + by^2 = 1 \qquad \dots \qquad \dots \qquad (1)$$

and let the line cut the plane at an angle α at (h, k) and let the orthogonal projection of the line be $x = h + r\cos\beta$, $y = k + r\sin\beta$, r being the distance of any point on it from (h, k). Then any point P on the line is

$$x = h + r\cos\beta\,;\ y = k + r\sin\beta\,;\ z = r\tan\alpha.$$

Let Q (x, y) on (1) be nearest to P, so that

$$[(h + r\cos\beta - x)^2 + (k + r\sin\beta - y)^2 + r^2\tan^2\alpha]$$

is least : the condition for which is easily seen to be

$$\frac{h + r\cos\beta - x}{ax} = \frac{k + r\sin\beta - y}{by}. \qquad \dots \qquad (2)$$

Any point on PQ is

$$\left.\begin{aligned} X &= \frac{h + r\cos\beta + \lambda x}{\lambda + 1}, \\ Y &= \frac{k + r\sin\beta + \lambda y}{\lambda + 1}, \\ Z &= \frac{r\tan\alpha}{\lambda + 1}. \end{aligned}\right\} \qquad \dots \qquad \dots \qquad (3)$$

Elimination of x, y, r, λ from (1), (2), (3) leads to the required result.

Case (ii) Parabola : Take its equation as $y^2 = 4ax$. $\qquad \dots \qquad (1')$

Then (2) becomes $\dfrac{h + r\cos\beta - x}{2a} + \dfrac{k + r\sin\beta - y}{y} = 0. \dots \qquad (2')$

Elimination from (1′), (2′), (3) leads to the result.

Case (iii) Circles : Q may be taken as $(a \cos \theta,\ a \sin \theta)$, k may be made zero. Here (2) reduces to

$$\frac{h + r\cos\beta}{\cos\theta} = \frac{r \sin\beta}{\sin\theta} = \mu \text{ (say)}, \qquad \ldots \quad (2'')$$

and (3) becomes

$$X = \frac{h + r\cos\beta + \lambda a \cos\theta}{\lambda + 1} = \left(\frac{\mu + \lambda a}{\lambda + 1}\right)\cos\theta, \quad \ldots \quad (3'')$$

$$Y = \frac{r\sin\beta + \lambda a \sin\theta}{\lambda + 1} = \left(\frac{\mu + \lambda a}{\lambda + 1}\right)\sin\theta, \quad \ldots \quad (4'')$$

$$Z = \frac{r \tan\alpha}{\lambda + 1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5'')$$

From (2″) and (3″)

$$\frac{h + r\cos\beta}{X} = \frac{r\sin\beta}{Y} = \frac{h\sin\beta}{X\sin\beta - Y\cos\beta}; \quad \ldots \quad (6'')$$

(5″) now gives

$$\lambda + 1 = \frac{hy \tan\alpha}{Z(X\sin\beta - Y\cos\beta)}; \quad \ldots \quad \ldots \quad (7'')$$

(2″) gives

$$\left.\begin{aligned} \mu\cos\theta &= \frac{h\sin\beta}{X\sin\beta - Y\cos\beta}\cdot X \\ \mu\sin\theta &= \frac{h\sin\beta}{X\sin\beta - Y\cos\beta}\cdot Y \end{aligned}\right\} \text{by using } (6'') \quad \ldots \quad (8'')$$

(3″) and (4″) become

$$(\lambda + 1)X - \mu\cos\theta = \frac{h}{Z}\left(\frac{Y\tan\alpha - Z\sin\beta}{X\sin\beta - Y\cos\beta}\right) X = \lambda a \cos\theta. \quad (9'')$$

$$(\lambda + 1)Y - \mu\sin\theta = \frac{h}{Z}\left(\frac{Y\tan\alpha - Z\sin\beta}{X\sin\beta - Y\cos\beta}\right) Y = \lambda a \sin\theta. \quad (10'')$$

Squaring (9″) and (10″) and taking 'λ' from (7″) the surface traced out is

$$h^2(Y\tan\alpha - Z\sin\beta)^2(X^2 + Y^2) = [hY\tan\alpha - Z(X\sin\beta - Y\cos\beta)]^2 a^2.$$

Question 1239.

(R. VAIDYANATHASWAMY):—If P, Q, R, S be four points in an Argand diagram, P′, Q′, R′, S′ their inverses with respect to a real or imaginary circle, show that the cross ratios (PQRS) and (P′Q′R′S′) are conjugate complex numbers.

Solution by S. Audinarayanan.

Suppose the points P, Q, R, S to represent the complex numbers $x_1 + iy_1$, $x_2 + iy_2$, $x_3 + iy_3$ and $x_4 + iy_4$ in the Argand diagram. Now the cross-ratio (PQRS)

$$= \frac{[\overline{x_1 - x_2} + i\,\overline{y_1 - y_2}]\,[\overline{x_3 - x_4} + i\,\overline{y_3 - y_4}]}{[\overline{x_1 - x_4} + i\,\overline{y_1 - y_4}]\,[\overline{x_3 - x_2} + i\,\overline{y_3 - y_2}]}$$

$$= \frac{(a + bi)\,(c + di)}{(e + fi)\,(g + hi)}$$

$$= \frac{(ac - bd) + i(bc + ad)}{(eg - hf) + i(fg + ch)} = \frac{x + iy}{z + iw} \text{ say,}$$

$$= \frac{xz + yw + i(yz - xw)}{z^2 + w^2}.$$

We know that the inverse of a point is the reflection of the geometrical inverse with respect to the x-axis. The geometrical inverse of P, Q, R, S are points in the Argand diagram representing numbers

$$\frac{k^2}{x_1 + iy_1},\ \frac{k^2}{x_2 + iy_2},\ \frac{k^2}{x_3 + iy_3},\ \frac{k^2}{x_4 + iy_4}.$$

The numbers represented by the true inverse points will have the sign of their x's opposite to those of the geometrical inverse points.

Hence the numbers represented by the inverse points P′, Q′, R′, S′ are

$$\frac{k^2}{-x_1 + iy_1},\ \frac{k^2}{-x_2 + iy_2},\ \frac{k^2}{-x_3 + iy_3},\ \frac{k^2}{-x_4 + iy_4}.$$

Now the cross ratio (P′Q′R′S′)

$$= \frac{\left(\dfrac{1}{x_2 - iy_2} - \dfrac{1}{x_1 - iy_1}\right)\left(\dfrac{1}{x_4 - iy_4} - \dfrac{1}{x_3 - iy_3}\right)}{\left(\dfrac{1}{x_2 - iy_2} - \dfrac{1}{x_3 - iy_3}\right)\left(\dfrac{1}{x_4 - iy_4} - \dfrac{1}{x_1 - iy_1}\right)}$$

$$= \frac{[\overline{x_1 - x_2} - i\,\overline{y_1 - y_2}]\,[\overline{x_3 - x_4} - i\,\overline{y_3 - y_4}]}{[\overline{x_3 - x_2} - i\,\overline{y_3 - y_2}]\,[\overline{x_1 - x_4} - i\,\overline{y_1 - y_4}]}$$

$$= \frac{(a - bi)\,(c - di)}{(g - hi)\,(e - fi)}$$

$$= \frac{(ac - bd) - i(bc + ad)}{(eg - hf) - i(fg + ch)} = \frac{x - iy}{z - iw}$$

$$= \frac{xz + yw - i(yz - xw)}{z^2 + w^2}$$

= conjugate of (PQRS).

Thus it is established that the cross ratios (PQRS) and (P′Q′R′S′) are conjugate complex numbers.

Question 1251.

(C. KRISHNAMACHARI AND M. BHIMASENA RAO) :—Let $[A_r, A_n]$ stand for the persymmetric determinant

$$\begin{vmatrix} A_r, & A_{r+1}, & A_{r+2}, & \ldots\ldots & A_n \\ A_{r+1}, & A_{r+2}, & A_{r+3}, & \ldots\ldots & A_{n+1} \\ A_{r+2}, & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_n, & A_{n+1}, & & & A_{2n-r} \end{vmatrix}$$

Then show that :

(*a*) If $A_s = 1.3.5 \ldots\ldots (2s-1)$,
$[A_r, A_n] = A_r . A_{r+1} \ldots\ldots A_n . 2^{n-r} . 4^{n-r-1} . 6^{n-r-2} \ldots\ldots (2n-2r)$.

(*b*) If $A_s = s\,!$,
$[A_r, A_n] = A_r A_{r+1} \ldots\ldots A_n \cdot 1!.2!.3! \ldots\ldots (n-r)!$

(*c*) If $A_s = m(m+1)(m+2) \ldots\ldots (m+s-1)$,
$[A_r, A_n] = A_r . A_{r+1} \ldots\ldots A_n . 1!.2!.3! \ldots\ldots (n-r)!$

(*d*) The value of the determinant

$$\begin{vmatrix} 1 & 1 & \ldots\ldots & 1 \\ m+r & m+r+1 & \ldots\ldots & m+n \\ (m+r)_2 & (m+r+1)_2 & \ldots\ldots & (m+n)_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ (m+r)_{n-r} & (m+r+1)_{n-r} & \ldots\ldots & (m+n)_{n-r} \end{vmatrix}$$

is independent of m, where x_p denotes

$$x(x+1)(x+2) \ldots\ldots (x+p-1).$$

Solution by Martyn Thomas.

The mode of evaluation of the several determinants is the same for all, namely, by first rewriting them with the elements of the several columns beginning with those of the first column, so that after removal of the common factors $A_r, A_{r+1} \ldots\ldots A_n$, and changing every row but the first one into the difference between that row and its preceding one, the common factors may be removed again, and the process may be repeated, the order of the determinants being lowered thereby.

(*a*) $A_r = 1.3.5 \ldots\ldots (2r-1)$.

$$\therefore \Delta = \begin{vmatrix} A_r, & (2r+1)A_r, & (2r+1)(2r+3)A_r, & \ldots\ldots & (2r+1)(2r+3)\ldots\ldots(2n-1)A_r \\ A_{r+1}, & (2r+3)A_{r+1}, & (2r+3)(2r+5)A_{r+1}, & \ldots\ldots & (2r+3)(2r+5)\ldots\ldots(2n+1)\ A_{r+1} \\ A_n, & (2n+1)(2n+1)A_n, & \ldots & & (2n+1)\ldots\ldots(4n-2r-1)A_n \end{vmatrix}$$

$$=(A_rA_{r+1}\ldots A_n)\begin{vmatrix} 1, & (2r+1), & \ldots\ldots\ldots & \\ 0, & 2, & (2r+3)4, & (2r+3)\ldots(2n-1)(2n-2r) \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 2, & (2n+1)4, & (2n+1)\ldots\ldots(2n-2r) \end{vmatrix}$$

$$=(A_rA_{r+1}..A_n).2.4.6..(2n-2r)\begin{vmatrix} 1, & 2r+3, & \ldots\ldots\ldots & \\ 0. & 2, & (2r+5)4, & \ldots(2r+5)\ldots(2n-2r-2) \\ 0, & 2, & (2n+3)4, & \ldots(2n+3)\ldots(2n-2r-2) \end{vmatrix}$$

$$= (A_rA_{r+1}\ldots\ldots A_n)\,[2.4.6\ldots\ldots(2n-2r)]\,[2.4.6\ldots\ldots(2n-2r-2)]\times$$
$$[2.4.6\ldots\ldots(2n-2r-4)]\ldots\ldots[2.4.6]\,[2.4]\,[2]$$

$$= (A_rA_{r+1}\ldots\ldots A_n).\,2^{n-r}.\,4^{n-r-1}.\,6^{n-r-2}\ldots\ldots(2n-2r-2)^2\,(2n-2r).$$

(*b*) Here $A_r = r\,!$.

$$\Delta=\begin{vmatrix} A_r, & (r+1)A_r, & (r+1)(r+2)A_r, & \ldots(r+1)(r+2)\ldots n.A_r \\ A_{r+1}, & (r+2)A_{r+1}, & (r+2)(r+3)A_{r+1}, & \ldots(r+2)(r+3)\ldots(n+1)A_r \\ \ldots & \ldots & \ldots & \ldots \\ A_n, & (n+1)A_n, & \ldots\ldots\ldots & (n+1)(n+2)\ldots(2n-r)A_{r+1} \end{vmatrix}$$

$$=(A_rA_{r+1}\ldots A_n)\begin{vmatrix} 1, & r+1, & \ldots\ldots\ldots & \\ 0, & 1, & (r+2)2, & \ldots\ldots(r+2)\ldots\ldots n(n-r) \\ 0, & 1, & (n+1)2, & \ldots\ldots(n+1)\ldots\ldots(n-r) \end{vmatrix}$$

$$= (A_rA_{r+1}\ldots A_n)\,[1.2.3\ldots(n-r)]\,[1.2.3\ldots(n-r-1)]\times$$
$$[1.2.3\ldots(n-r-2)]\ldots[1.2.3]\,[1.2]\,[1]$$
$$= (A_rA_{r+1}\ldots A_n)(n-r)!\,(n-r-1)!\ldots 3!\,2!\,1!.$$

(*c*) Here $A_r = m(m+1)(m+2)\ldots\ldots(m+r-1)$.

$$\Delta=\begin{vmatrix} A_r,\ (m+r)A_r,\ (m+r)(m+r+1)A_r\ \ldots\ldots, \\ \qquad (m+r)\ldots\ldots(m+n-1)A_r \\ A_{r+1},\ (m+r+1)A_{r+1},\ (m+r+1)(m+r+2)A_{r+1}\ldots, \\ \qquad (m+r+1)\ldots\ldots(m+n)A_{r+1} \\ \ldots\ldots \\ A_n,\ (m+n)A_n,\ \ldots\ldots\ldots \end{vmatrix}$$

$$=(A_rA_{r+1}\ldots A_n)\begin{vmatrix} 1, & m+r, & \ldots\ldots\ldots & \\ 0 & 1, & (m+r+1)2 & \ldots\ldots(m+r+1)\ldots(n-r) \\ \ldots & & & \\ 0, & 1, & (m+n)2 & \ldots\ldots\ldots \end{vmatrix}$$

$$=(A_rA_{r+1}\ldots A_n)\,[1.2.3\ldots(n-r)]\,[1.2.3\ldots(n-r-1)]\ldots[1.2.3]\,[1.2]\,[1]$$
$$=(A_rA_{r+1}\ldots A_n)(n-r)!\,(n-r-1)!\ldots 3!\,2!\,1!.$$

(*d*) Here $(m+r)_p = (m+r)(m+r+1)\cdots$ to p factors.

$$\Delta = \begin{vmatrix} 1, & 1 & \cdots\cdots & 1 \\ m+r, & m+r+1 & & m+n \\ (m+r)(m+r+1), & (m+r+1)(m+r+2), & \cdots\cdots & (m+n),(m+n+1) \\ \cdots & \cdots & \cdots & \cdots \\ (m+r)\cdots(m+n-1), & (m+r+1)\cdots(m+n), & \cdots & (m+n)\cdots(m+2n-r-1) \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0, & \cdots\cdots \\ m+r, & 1, & 1, & \cdots\cdots \\ 1 & (m+r+1)2, & (m+r+2), & \cdots\cdots \\ 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & (m+r+1)\ldots(n-r), & (m+r+2)\ldots(n-r), & \end{vmatrix}$$

$$= [1.2.3 \cdots\cdots (n-r)]\,[1.2.3 \cdots\cdots (n-r-1)] \cdots\cdots [1.2.3][1.2][1]$$

$$= (n-r)!\,(n-r-1)! \cdots\cdots 3!\,2!\,1!,$$

which is independent of m.

Question 1259.

(P. V. Seshu Iyer) :—(1) Show that the values of x for which the function $u = xf(x)$ is a maximum are the abscissæ of the points where $y = f(x)$ touches a member of the hyperbola $xy = c$.

(2) If $y = f(x)$ touches a member of the family $xy = c$ at a point whose abscissa is x, show that the curve $y = f(x) - \frac{k}{x}$, touches another of the family at a point of the same abscissa x, and find the constant c of that other member.

Solution by P. Krishnamachari, S. Audinarayanan, I. Totadri Iyengar, C. Ranganathan and V. A. Mahalingam.

(1) The values of x for which the function $u = xf(x)$ is a maximum are given by the equation $\frac{du}{dx} = 0,$

i.e., by
$$f(x) + xf'(x) = 0. \qquad \ldots \qquad \ldots \quad (1)$$

The abscissæ of the points of intersection of $y = f(x)$ with $xy = c$ are given by
$$xf(x) = c, \qquad \ldots \qquad \ldots \qquad \ldots \quad (2)$$

and if the two curves should also touch at this point
$$f'(x) = -\frac{c}{x^2}. \qquad \ldots \qquad \ldots \quad (3)$$

The eliminant of (2) and (3) gives the abscissæ of the point of contact of $y = f(x)$ with the family of curves $xy = c$; and this is

$$x[f(x) + xf'(x)] = 0.$$

Rejecting the factor x, we see that the first part follows.

(2) If c_1 is the 'c' of the first member and c_2 that of the second, then from the above $\quad x f(x) = c_1$;

and in the latter case $\quad x f(x) - k = c_2, \quad \ldots \quad \ldots \quad (4)$

and $\quad f'(x) - \dfrac{k}{x^2} = -\dfrac{c_2}{x^2}. \quad \ldots \quad \ldots \quad (5)$

Eliminating c_2 from (4) and (5), we get $f(x) + xf'(x) = 0$, which shows that x can be the abscissa of the point of contact of $y = f(x) - \dfrac{k}{x}$ with a member of the family of rectangular hyperbolas.

From (4), we see that $c_2 = c_1 - k$, which gives the 'c' of the latter member in terms of that of the former.

Question 1265.

(A. A. Krishnaswamy Iyengar) :—Show that every odd number can be expressed as the sum of seven squares (excepting 1, 3, 5, 9, 11, 17).

Solution by I. Totadri Iyengar.

Lemma :—If $3\,M = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$ (in which of course either only one of them say α or all of them should be multiples of 3), then $M = \alpha_1^{\,2} + \beta_1^{\,2} + \gamma_1^{\,2} + \delta_1^{\,2}$,

where

$$\left.\begin{aligned} \alpha_1 &= \frac{\sqrt{\beta^2} + \sqrt{\gamma^2} + \sqrt{\delta^2}}{3} \\ \beta_1 &= \frac{\alpha + \sqrt{\beta^2} - \sqrt{\gamma^2}}{3} \\ \gamma_1 &= \frac{\alpha + \sqrt{\gamma^2} - \sqrt{\delta^2}}{3} \\ \delta_1 &= \frac{\alpha + \sqrt{\delta^2} - \sqrt{\beta^2}}{3}. \end{aligned}\right\}$$

To the absolute value of the radical that sign is to be attached which makes $\sqrt{\beta^2} \equiv \sqrt{\gamma^2} \equiv \sqrt{\delta^2} \pmod{3}$.

Let the given number n be of the form $4\lambda + 1$, and let p be any prime contained in the form $3^{2x+1}\,n - 2$.

Evidently $p \equiv 1 \bmod. 4.$

$\therefore \qquad p = \alpha^2 + \beta^2.$

Then $3^{2x+1} . n = \alpha^2 + \beta^2 + 1^2 + 1^2.$

So that, by the Lemma, we can express n as the sum of 4 squares.

Let finally therefore $n = a^2 + b^2 + c^2 + d^2$ of which if a, b, c, d are not all multiples of 3, a is the only multiple of 3.

Then a^2 is of the form $4n + 1$ and hence can be split up as above into the sum of 4 squares, say

$$a^2 = e^2 + f^2 + g^2 + h^2.$$

So we finally get

$$n = b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2$$

as the solution.

If n is of the form $4\lambda + 3$, we take p to be the prime contained in the form $3^{2x} \cdot n - 2$ and proceed as before.

In the special cases mentioned some of the numbers b, c, d, etc., vanish.

N. B. With certain exceptions the same theorem holds also in the case of the even numbers, (*Vide* Sylvester: *Collected Works*, Vol. II, p. 102.) In general, we can say that all numbers can be expressed as the sum of $(3n + 1)$ squares.

Question 1271.

(K. Satyanarayana) :—(i) p being any odd prime, prove that

$$2^{p-1} - 1 \not\equiv 0 \bmod. (p^2.)$$

Hence or otherwise, establish that the numerator of

$$\left\{1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots\ldots\ldots + (-1)^{\frac{p+1}{2}} \frac{1}{\frac{1}{2}(p-1)}\right\} \not\equiv 0 \bmod. p.$$

(ii) p being a prime greater than 3, prove that,

$$\left\{(p-1)! - (-1)^{\frac{p-1}{2}} \left[2^{p-1} \left(\frac{p-1}{2}\right)!\right]^2\right\} \equiv 0 \bmod. p^3$$

Solution by I. Totadri Iyengar.

(i) Let $p = 2n + 1$. If it be possible, let

$$2^{p-1} \equiv 1 \text{ mod. } p^2.$$

Then $$2^{2p-2} = 2^{4n} \equiv 1 \text{ mod. } p^2$$

or $$\frac{2^{4n}-1}{n} \equiv 0 \text{ mod. } p.$$

Hence from Q, 1247 already solved [*J. I. M. S.*, XV, p 60.]

$$a \equiv (-1)^n b \text{ mod. } p.$$

i. e., $$\frac{(2n)!+1}{p} \equiv (-1)^n . \frac{n!+(-1)^n}{p} \text{ mod. } p$$

i.e., $$(2n)! \equiv (-1)^n n! \text{ mod. } p^2$$

Hence $(p-1)(p-2)\ldots(p-n).\, n! \equiv (-1)^n\, n! \text{ mod. } p^2$.

$$\therefore (-1)^n \left\{ n! - n!\, p \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n}\right) \right\} \equiv (-1)^n \text{ mod. } p^2.$$

Now from the solution to Q. 1197, since $2^{4n} - 1 \equiv 0$ mod. p^2

$$\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{n}\right) n! \equiv 0 \text{ mod. } p.$$

$\therefore$ $\quad n! \equiv 0$ mod. p^2 which is impossible.

Hence $$2^{p-1} - 1 \not\equiv 0 \text{ mod. } p^2.$$

Now $$2^{p-1} - 1 = (p-1) + \frac{(p-1)(p-2)}{1.2} + \ldots\ldots$$

$$\frac{(p-1)(p-2)}{1.2} + (p-1) + 1$$

$$= p + p.\frac{p-1}{1.2} + p\frac{(p-1)(p-2)}{1.2.3} + \ldots$$

on adding the 1st and the last; the 2nd and the last but one, etc.

Simplifying we have the numerator of

$$\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots\ldots + (-1)^{\frac{p+1}{2}} \frac{1}{n}\right) \not\equiv 0 \text{ mod. } p.$$

(ii) It is well-known from Lagrange that the numerator of

$$\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{p-1}\right) \equiv 0, \text{ mod. } p\,;$$

also that of

$$1 + \frac{1}{1.2} + \frac{1}{1.3} + \ldots\ldots + \frac{1}{2.3} + \ldots \equiv 0, \text{ mod. } p.$$

Hence on squaring the first and subtracting twice the second, we get

$$\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\,.\,+\frac{1}{(p-1)^2}\right\}\equiv 0 \text{ mod. } p.$$

But

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots\frac{1}{(p-1)^2}=\left\{1-\frac{1}{(p-1)^2}\right\}$$

$$+\left\{\frac{1}{2^2}-\frac{1}{(p-2)^2}\right\}+\left\{\frac{1}{3^2}-\frac{1}{(p-3)^2}\right\}+\ldots\ldots$$

$$\equiv 0 \text{ mod. } p.$$

$$\therefore \quad 1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots+\frac{1}{(p-2)^2}\equiv 0 \text{ mod. } p.$$

From the solution to Q. 1247

$$2^{2n}\,.\,2n\,!\equiv(-1)^n\,1^2\,.\,3^2\ldots\ldots(p-2)^2 \text{ Nr. of } \left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\right.$$

$$\left.+\frac{1}{p-2^2}\right)^2 \text{ mod. } p^3.$$

$$\equiv(-1)^n\,1^2\,.\,3^2\ldots\ldots(p-2)^2, \text{ mod. } p, \text{ from the above.}$$

Hence just as in Q. 1247, it follows that

$$2n\,!-(-1)^n\,2^{4n}\,(n\,!)^2\equiv 0, \text{ mod. } p^3,$$

which is the result required.

Questions for Solution.

1336. (MARTYN THOMAS):—An infinite straight line occupying the positive part of the axis of z attracts according to the law of nature. Show, from elementary considerations, that its potential is a function of $(r-z)$, and hence, or otherwise, prove that the potential is $A \log (r-z)+B$, where A and B are constants.

1337. (A. NARASINGA RAO):—a_1, a_2 are a pair of straight lines through A and b_1, b_2 another pair through B. Show that

(i) through a given point x can be drawn in general a pair of lines x_1, x_2 in two different ways so that the quadrilaterals $a_1\, a_2\, x_1\, x_2$ and $b_1\, b_2\, x_1\, x_2$ are circumscribed to circles.

(ii) if c_1 c_2 be a third pair of lines through C, and if the quadrilaterals $a_1\, a_2\, x_1\, x_2$, $b_1\, b_2\, x_1\, x_2$, $c_1\, c_2\, x_1\, x_2$ are circumscribed to circles, the common point X of x_1 and x_2 must lie on one of four onics in the plane.

1338. SELECTED—(Prof. CAYLEY):—Show that the surfaces

$$xyz = 1, \quad yz + zx + xy + x + y + z + 3 = 0$$

intersect in two distinct cubic curves; and find the equations of the cubic cones which have their vertices at the origin and pass through these curves respectively.

1339. (R. VAIDYANATHASWAMY):—A quadratic form $\Sigma a_{ij}\, x_i\, x_j$ $(i, j = 1, 2, \ldots n)$ is such that

$$a_{ji} a_{jk} a_{ik} = 0, \ a_{ij} = a_{ji}$$

Show that by permuting the variables it can be reduced to the form

$$2\,(a_{12}\, x_1\, x_2 + a_{23}\, x_2\, x_3 + a_{34}\, x_3\, x_4 + \ldots\ldots + a_{n-1,\,n}\, x_{n-1}\, x_n)$$

1340. (R. GOPALASWAMI):—A conic cuts the sides of a quadrilateral at eight points. If two of these points lying on different sides be joined to meet the two other sides, two new points are obtained. Show that the eight new points obtained (by choosing four pairs in any manner whatever) lie on a conic.

1341. (R. GOPALASWAMI):—A_r $(r = 1, 2, 3, 4)$ are four points on the sides of a quadrilateral. A conic through A_r cuts the sides at four other points B_r. A conic through B_r cuts them at C_r and so on the tetrads D_r, E_r, F_r are obtained. Show that any two of the tetrads A_r, B_r, C_r, D_r, E_r, F_r lie on a conic. The conic $(A_r\, B_r)$ meets $(D_r\, E_r)$ at points P_r. $(B_r\, C_r)$ meets $(E_r\, F_r)$ at Q_r. $(C_r\, D_r)$ meets $(F_r\, A_r)$ at R_r. Show that the twelve points P_r, Q_r, R_r lie on a conic.

The above is an extension of Pascal's theorem. Extend it further.

1342. (V. RAMASWAMI AIYAR):— Prove that

$$\begin{vmatrix} 0 & x & x' & 0 & a & a' \\ 0 & x' & x & 0 & a' & a \\ y' & 0 & y & b' & 0 & b \\ y & 0 & y' & b & 0 & b' \\ z & z' & 0 & c & c' & 0 \\ z' & z & 0 & c' & c & 0 \end{vmatrix} = \begin{vmatrix} x^2 - x'^2, & y^2 - y'^2, & z^2 - z'^2 \\ a^2 - a'^2, & b^2 - b'^2, & c^2 - c'^2 \\ ax - a'x', & by - b'y', & cz - c'z' \end{vmatrix}$$

1343. (V. RAMASWAMI AIYAR):—Given a triangle ABC, if P be a variable point on a fixed diameter of its circumcircle that, we know its pedal circle passes through a fixed point Ω. Prove that the rectangular hyperbola passing through P and the feet of the perpendiculars drawn therefrom to the sides cuts the pedal circle of P again at the point diametrically opposite to Ω.

1344. (V. TIRUVENKATACHARYA):—Show that if $|q| < 1$,

(i) $(1-q^4)^2(1-q^8)(1-q^{12})^2(1-q^{16})\ldots\ldots = 1 - 2q^4 + 2q^{16} - \ldots$

(ii) $[(1-q^2)(1-q^6)(1-q^{10})\ldots\ldots]^{-1} = (1+q^2)(1+q^4)(1+q^6)\ldots.$

1345. (F. H. V. GULASEKHARAN):—If each of the ratios

$$(x_p, y_p, z_p, t_p), (x_q, y_q, z_q, t_q)$$

satisfies simultaneously the equations

$$\left.\begin{aligned} u_1x + u_2y + u_3z + u_4t &= 0 \\ v_1x + v_2y + v_3z + v_4t &= 0 \end{aligned}\right\}$$

and if

$$S_{pq} = \Sigma\, ax_px_q + \Sigma f(y_pz_q + y_qz_p),$$

the summation extending to the variables x, y, z, t and the co-efficients $a, b, c, d, f, g, h, l, m, n$; and if S'_{pq} denotes a similar function with co-efficients a', b', c' etc., prove that

$$\frac{S_{pp}S_{qq} - S_{pq}{}^2}{\Delta} = \frac{S_{pp}S'_{qq} + S_{qq}S'_{pp} - 2S_{pq}S'_{pq}}{\phi} = \frac{S'_{pp}S'_{qq} - S'^2_{pq}}{\Delta'},$$

where Δ, ϕ, Δ' are the co-efficients of k^2, k, k^0 in the equation

$$\begin{vmatrix} ka + a', & kh + h', & kg + g', & kl + l', & u_1, & v_1 \\ kh + h', & kb + b', & kf + f', & km + m', & u_2, & v_2 \\ kg + g', & kf + f', & kc + c', & kn + n', & u_3, & v_3 \\ kl + l', & km + m', & kn + n', & kd + d', & u_4, & v_4 \\ u_1, & u_2, & u_3, & u_4, & 0, & 0 \\ v_1, & v_2, & v_3, & v_4, & 0, & 0 \end{vmatrix} = 0.$$

Explain the results geometrically.

[The extension to the case of n variables satisfying $(n-2)$ linear equations is obvious.]

185293

LIST OF JOURNALS RECEIVED

(From the 2nd of June to the 2nd of August 1924).

1 Nature, Vol. 113, Nos. 2843, 2844, 2845, 2846, 2847, 2848, 2849, 2850, 2851.

2 Popular Astronomy, Vol. 32, Nos. 5, 6.

3 Philosophical Magazine, Vol. 47, Nos. 281 and 282, Vol. 48, No. 283.

4 Bulletin of the American Mathematical Society, Vol. 30, Nos. 3, 4, 5 and 6.

5 The American Mathematical Monthly, Vol. 31, Nos. 4, 5.

6 Proceedings of the Royal Society, Vol. 105, Nos. 733, 734.

7 Monthly Notices of the Royal Astronomical Society, Vol. 84 Nos. 5, 6.

8 The Astrophysical Journal, Vol. 59, Nos. 2, 3, 4.

9 Annals of Mathematics, Vol. 25, No. 1.

10 The Messenger of Mathematics, Vol. 53, No. 9.

11 Journal für die reine und Angewandte Mathematik, Band 53, Heft $\frac{3}{4}$.

12 Philosophical Transactions of the Royal Society of London, Series A, Vol. 224, Nos. 618, 619, 620.

13 Transactions of the American Mathematical Society, Vol. 26, No. 1.

14 Proceedings of the London Mathematical Society, Vol. 23, Part I.

15 The Mathematical Gazette, Vol. 12, No. 170.

16 Revista De Mathematicas Ano. V, Nos. 11 and 12 (59 & 60).

17 The Journal of the Science Association, Maharajah's College, Vizianagram, Vol. I, No. 4, Vol. II, No. 1.

18 Proceedings of the Edinburgh Mathematical Society, Vol. 42, (Session 1923-24), Part I.

19 Rendiconti del Circolo Matematico di Palermo, Tome XLVIII Fascicolo 1 (1924).

The Indian Mathematical Society

(Founded in 1907 for the Advancement of Mathematical Study and Research in India.)

THE COMMITTEE.

President:

BALAKRAM, M..A, I.C.S., Judge, Sholapur.

Treasurer:

S. NARAYANA AIYAR, M.A., Chief Accountant, Port Trust, Madras.

Librarian:

V. B. NAIK, M.A., Professor of Mathematics, Fergusson College, Poona.

Secretaries:

P. V. SESHU AIYAR, B.A., L.T., Principal, Government College, Kumbakonam.

M. T. NARANIENGAR, M.A., Professor of Mathematics, Central College, Bangalore.

Additional Members:

E. B. ROSS, M.A., Professor of Mathematics, Christian College, Madras.

D. D. KAPADIA, M.A., B.Sc., Professor of Mathematics, Deccan College, Poona.

HEMRAJ, M.A., Professor of Mathematics, Dyal Singh College, Lahore.

K. ANANDA RAO, M.A., Professor of Pure Mathematics, Presidency College, Madras.

S. V. RAMAMURTHY, M.A., I.C.S., Ramnad.

N. M. SHAH, B.A. (Cantab), Professor of Mathematics, New Poona College, Poona.

REV. A. STEICHEN, Ph.D., S.J., Professor of Physics, St. Xavier's College, Bombay.

OFFICERS OUTSIDE THE COMMITTEE.

Assistant Secretary:

G. A. Srinivasan, M.A., Assistant Professor of Mathematics, Presidency College, Madras.

Assistant Librarian:

V. A. APTE, M.A., Professor of Mathematics, Fergusson College, Poona.

Associates:

K. B. MADHAVA, M.A.
S. A. MANI, M.A., L.T.
A. NARASINGA ROW, M.A., L.T.
S. R. RANGANATHAN, M. A., L.T.

Head-Quarters: Poona.